Math 53, Discussions 116 and 118 **2022-09-09**

Conic sections and intro to 3D and vectors

Answers included

Questions

Question 1. What does the equation $y^2 = 4$ describe in $\mathbb{R}^2$? What about $\mathbb{R}^3$?

Question 2. If $\mathbf{r} = \langle x, y \rangle$, $\mathbf{a} = \langle a_1, a_2 \rangle$, and $\mathbf{b} = \langle b_1, b_2 \rangle$ (where a_1, a_2, b_1, b_2 are constants), expand out the equation

$$(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$$

and say what kind of shape it is.

Question 3. Can you express the magnitude (length) of a vector $\mathbf{v}$ in terms of the scalar (dot) product?

Question 4. Do the surfaces defined by the equations

$$x^2 + y^2 + (z-1)^2 = 25$$

and

$$x^2 + y^2 + z^2 = 9$$

intersect?

Question 5. Suppose that H_1 and H_2 are two planes in $\mathbb{R}^3$ (3-dimensional space). Which of the following might be the intersection $H_1 \cap H_2$? There are multiple correct answers.

(a) A plane.
(b) A line.
(c) A point.
(d) Empty (the planes don't intersect).

Question 6. Identify the following shapes in $\mathbb{R}^2$. Just a simple verbal description is fine.

(a) $4x^2 - 12x - 9y^2 - 6y + 7 = 0$
(b) $4x^2 - 12x - 9y^2 - 6y + 8 = 0$
(c) $4x^2 - 12x - 9y^2 - 6y + 9 = 0$

Question 7. Consider the line L with parametric equations

$$x = 3 + 3t, \qquad y = 2 - t, \qquad z = 5t$$

and the point $P(1, -2, 2)$. Find the point Q on the line L which minimizes the distance $|PQ|$, and say what this minimum distance is.

Below are brief answers to the worksheet exercises. If you would like a more detailed solution, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to questions

Question 1. In $\mathbb{R}^2$, a pair of lines. In $\mathbb{R}^3$, a pair of planes. I drew pictures in class.

Question 2. The equation is

$$\langle x - a_1, y - a_2 \rangle \cdot \langle x - b_1, y - b_2 \rangle = 0$$

which we can expand as

$$x^2 - (a_1 + b_1)x + a_1 b_1 + y^2 - (a_2 + b_2)y + a_2 b_2 = 0.$$

After completing the square, you will find that this is a circle.

Question 3. $\|\mathbf{v}\| = (\mathbf{v} \cdot \mathbf{v})^{1/2}$.

Question 4. No. I demonstrated this both algebraically and geometrically. The first is a sphere of radius 5 centered at $(0, 0, 1)$. The second is a sphere of radius 3 centered at $(0, 0, 0)$. The latter sphere is completely contained inside the former; they do not touch.

Question 5. All of these are possible except for the case of a point. However this is difficult to show (the purpose of the exercise was just to have you practice visualizing 3D).

(a) Yes, if the two planes completely coincide.
(b) Yes, this is the most common situation in fact.
(c) No, this is impossible.
(d) Yes, if the two planes are parallel.

Question 6.

(a) Hyperbola
(b) Pair of intersecting lines
(c) Hyperbola

Question 7. We will thoroughly revisit this question later.